AGEOSTROPHIC THEORY OF ZONAL FLOW ACCELERATION AND WAVE-ACTIVITY CONSERVATION LAW*

RAN Lingkun (冉令坤) and *GAO Shouting* (高守亭)

Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029

Received July 29, 2003; revised January 16, 2004

ABSTRACT

Motivated by ageostrophic interactions of wave and basic flow, the generalized relationships between 3-dimensional Eliassen-Palm flux and basic flows, which are suitable for small-amplitude and finite-amplitude disturbances, are derived. The local area-averaged and density-weighted mean flows are chosen as the basic flows. Under the assumption that the steady basic flows vary slowly in time and space, a quasi-conservative law of small amplitude wave activity is derived from Ertel's potential vorticity equation in isentropic coordinates.

The expressions of the new 3-D Eliassen-Palm flux and wave activity are presented in terms of Eulerian quantities so that they can be readily calculated by using observation data or model output data.

Key words: Eliassen-Palm flux, small-amplitude disturbances, finite-amplitude disturbances, wave-activity conservation law

I. INTRODUCTION

Since Eliassen and Palm (1961) firstly proposed the concept of Eliassen-Palm (E-P) flux, and Andrews and McIntyre (1976) introduced the residual circulation into the transformed Eulerian diagnostics, many meteorologists have conducted extensive and thorough studies on the wave-mean flow interaction and great progress has been made in understanding the forcing effects of waves or eddies on the evolution of basic flow (Andrews 1983; Andrews and McIntyre 1976; Gao et al. 1990; Pfeffer 1992; Plumb 1986; Takaya 2001).

In large-scale geophysical fluid dynamics, the E-P flux and wave-activity conservation law are dominantly important tools to investigate the evolution and propagation of transient waves in observational or model atmosphere (Edmon et al. 1980; Dunkerton et al. 1981). In order to gain the essential physical insights into the circulation evolution, the geostrophic or quasi-geostrophic assumptions are usually exploited in some studies of wave-mean flow interaction. The assumptions may be helpful to simplify the problem and capture the physical nature. However, restrictions of the geostrophic and quasi-geostrophic assumptions in practical applications are also evident. The geostrophic or quai-geostrophic assumptions applicable in the middle latitudes do not hold at and near the equator. The E-P theory and wave-activity conservative relation based on the assumptions

* This work was supported by the National Natural Science Foundation of China under Grant No. 40175017 and the Innovation Project of Chinese Academy of Sciences under Grant No. KZCX3-SW-217.

have singular values at the equator and are not able to interpret the cross-equator propagations of wavetrains between the Northern and Southern Hemispheres. Although there is some consensus that the mean zonal velocity is in geostrophic balance, there is no a justification for applying the quasi-geostrophic approximation to the zonally averaged north-south velocity. In addition, the inertia-gravity waves as an important component of forcing for some processes are filtered out in the geostrophic or quasi-geostrophic theories.

Contrasted with the geostrophic or quasi-geostrophic theories, the primitive equations in ageotrophic dynamic framework can provide more profound insights into the actual processes of wave-basic flow interactions. Tung (1986) thought that it is important to use isentropic coordinates in the formulation of an ageostrophic theory of zonally averaged circulation because the role played by the eddy E-P flux divergence in the forcing of residual zonal mean circulation becomes more difficult to understand in pressure coordinates due to the presence of mean ageostrophic meridional circulations. However, such a problem does not appear in isentropic coordinates. In isentropic coordinates the dissipative forcing can be easily separated from diabatic process and the governing equations reduce to be two-dimensional for adiabatic process. Due to these advantages, isentropic coordinates have been adopted by a number of authors. In the original studies, Andrews (1983) extended the study of Eliassen and Plum (1961) into finite-amplitude in isentropic coordinates, gained the corresponding transformed Eulerian equations and discussed the treatment method when the isentropic surfaces intersect the lower boundary. Haynes (1988) used the momentum-Casimir and energy-Casimir methods to study the finite-amplitude, local wave-activity relations for disturbances to zonal and nonzonal flows, taking account of the effects of forcing and dissipation. Tung (1986) formulated the ageostrophic theory of zonally averaged circulation using primitive equations in isentropic coordinates on the sphere. He gave the generalized definition of E-P flux divergence, and discussed the relation between E-P flux divergence and zonally averaged flow and the parameterization of E-P flux pseudo-divergence. Yang et al. (1990) performed a diagnostic study of E-P flux divergence and isentropic mixing coefficient in the stratosphere for different seasons and for both hemispheres.

Among the previous studies, the conventional E-P flux has been proven to be a powerful tool for diagnosing the propagation of Rossby waves and their interactions with zonal-mean flow on the meridional-vertical plane. The conventional E-P flux, a 2-D vector whose meridional and vertical components include zonally averaged eddy momentum and thermal fluxes, can represent the wave propagation on the meridional-vertical plane. However, it can not represent the propagation of waves in the zonal direction. This is the shortcoming of traditional E-P flux. If one is interested in the evolution of a locally forced wave packet that influences local weather and climate, its zonal propagation needs to be diagnosed explicitly. In some cases the zonal inhomogeneities of basic flow must be considered.

In this study, we attempt to generate the ageotrosphic theory of wave-basic flow interactions in isentropic coordinates, including the wave-activity conservation law and the relationships of 3-D E-P flux and basic flows. The local area-averaged zonal flow and density-weighted mean zonal flow are chosen as the basic flows so that the basic flows are functions of time and space. In Section II, the relations of the 3-D E-P flux and basic flow are derived. The wave-activity conservation law is discussed in Section III. A summary of the results is contained in Section IV.

II. THE GOVERNING EQUATIONS

In isentropic coordinates (x, y, θ), in which the vertical coordinate is defined by $\theta=T\left(\frac{p_0}{p}\right)^{R/c_p}$, the hydrostatic primitive equations may be written in the form as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\dot{\theta}\frac{\partial u}{\partial \theta}-fv=-\frac{\partial \varphi}{\partial x}+F, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\dot{\theta}\frac{\partial v}{\partial \theta}+fu=-\frac{\partial \varphi}{\partial y}+G, \tag{2}$$

$$\frac{\partial \varphi}{\partial \theta}=c_p\left(\frac{p}{p_0}\right)^{R/c_p}, \tag{3}$$

$$\frac{\partial \sigma}{\partial t}+\frac{\partial u\sigma}{\partial x}+\frac{\partial v\sigma}{\partial y}+\frac{\partial \dot{\theta}\sigma}{\partial \theta}=0, \tag{4}$$

$$\dot{\theta}=\frac{\theta}{c_pT}Q, \tag{5}$$

where $\sigma=-\frac{1}{g}\frac{\partial p}{\partial \theta}$ is the density in isentropic coordinates, $\varphi=c_pT+gz$ is the Montgomery potential, F and G are the body force per unit mass exerted on the fluid, and Q is the diabatic heating rate.

The zonal momentum equation is given by combining Eq. (1) and Eq. (4)

$$\frac{\partial \sigma u}{\partial t}+\frac{\partial u\sigma u}{\partial x}+\frac{\partial v\sigma u}{\partial y}+\frac{\partial \dot{\theta}\sigma u}{\partial \theta}-f\sigma v=-\sigma\frac{\partial \varphi}{\partial x}+\sigma F. \tag{6}$$

Following the hydrostatic equation (3) and the definition of density, it is ready to obtain

$$\sigma\frac{\partial \varphi}{\partial x}=-\frac{\partial}{\partial \theta}\left(\frac{p}{g}\frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{R}{R+c_p}\frac{p}{g}\frac{\partial \varphi}{\partial \theta}\right), \tag{7}$$

$$\sigma\frac{\partial \varphi}{\partial y}=-\frac{\partial}{\partial \theta}\left(\frac{p}{g}\frac{\partial \varphi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{R}{R+c_p}\frac{p}{g}\frac{\partial \varphi}{\partial \theta}\right). \tag{8}$$

Equations (6) and (7) may be combined to rewrite the zonal momentum equation as

$$\frac{\partial \sigma u}{\partial t}+\frac{\partial}{\partial x}\left(u\sigma u+\frac{R}{R+c_p}\frac{p}{g}\frac{\partial \varphi}{\partial \theta}\right)+\frac{\partial v\sigma u}{\partial y}+\frac{\partial}{\partial \theta}\left(-\frac{p}{g}\frac{\partial \varphi}{\partial x}\right)-f\sigma v=\sigma F-\frac{\partial \dot{\theta}\sigma u}{\partial \theta}. \tag{9}$$

In order to get 3-D E-P flux vector, we use local area averages instead of zonal averages. The local area average is defined by

$$\overline{(\cdot)}=\frac{1}{\Delta x\Delta y}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}(\cdot)\mathrm{d}x\mathrm{d}y, \tag{10}$$

where Δx and Δy are the spatial scales in x and y directions, respectively. They are generally specified a priori, relying on the motion scales that we are interested in. Thus the variables operated by Eq. (10) are still functions of time and space. It is convenient to separate the flows into two parts: averaged part and deviation part, such as

$$u=\bar{u}+u', \quad v=\bar{v}+v', \quad \sigma=\bar{\sigma}+\sigma', \quad p=\bar{p}+p', \quad \varphi=\bar{\varphi}+\varphi'.$$

The averaged part denoted by overbar "—" is referred to as the basic state, and deviation part denoted by prime "'" is the so-called "eddy" or "wave".

In isentropic coordinates, it is sometimes more useful to use density-weighted average which is defined as

$$\widetilde{(\cdot)} = \frac{\overline{\sigma(\cdot)}}{\overline{\sigma}} = \frac{\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\sigma(\cdot)\mathrm{d}x\mathrm{d}y}{\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\sigma\mathrm{d}x\mathrm{d}y}. \tag{11}$$

The deviation from the density-weighted average is denoted by asterisk, i. e. ,

$$u^* = u - \widetilde{u}, \quad v^* = v - \widetilde{v}, \quad \theta^* = \theta - \widetilde{\dot{\theta}}, \tag{12}$$

where $\widetilde{u} = \frac{\overline{\sigma u}}{\overline{\sigma}}$, $\widetilde{v} = \frac{\overline{\sigma v}}{\overline{\sigma}}$, $\widetilde{\dot{\theta}} = \frac{\overline{\sigma \dot{\theta}}}{\overline{\sigma}}$.

With β plane approximation, the averaged zonal momentum equation takes the following form

$$\frac{\partial(\overline{\sigma}\widetilde{u})}{\partial t} = -\nabla \cdot \boldsymbol{J} - \nabla \cdot \boldsymbol{F} + f\overline{\sigma}\widetilde{v} + \overline{Q}, \tag{13}$$

where

$$\boldsymbol{J} = \begin{pmatrix} \widetilde{u}\overline{\sigma}\widetilde{u} + \frac{R}{R+c_p}\frac{\overline{p}}{g}\frac{\partial\overline{\varphi}}{\partial\theta} \\ \widetilde{u}\overline{\sigma}\widetilde{v} \\ -\frac{\overline{p}}{g}\frac{\partial\overline{\varphi}}{\partial x} \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} \overline{u^*\sigma u^*} + \frac{1}{g}\frac{R}{R+c_p}\overline{p^*\frac{\partial\varphi^*}{\partial\theta}} \\ \overline{u^*\sigma v^*} \\ -\frac{1}{g}\overline{p'\frac{\partial\varphi'}{\partial x}} \end{pmatrix},$$

and

$$\overline{Q} = \overline{\sigma F} - \frac{\partial}{\partial\theta}(\widetilde{u}\overline{\sigma}\widetilde{\dot{\theta}} + \overline{u^*\sigma\dot{\theta}^*}).$$

It is clear from Eq. (13) that the tendency of averaged momentum is dependent on the combination of four parts: the basic state flux divergence, the eddy flux divergence, the Coriolis term associated with the earth's rotation effect, and the forcing effects such as friction force and diabatic heating. Here, we would like to emphasize the second term in the right-hand side of Eq. (13), a generalized version of the 3-D E-P flux divergence that represents the forcing influences of eddy transports on the basic flow.

According to the interpretations of 2-D E-P flux given by Andrews (1983) and Tung (1986), the 3-D E-P flux obtained here can be viewed as a 3-D force exerted by eddy on a thin tube, the meridional component and the first term in the zonal component of the 3-D E-P flux represent the lateral flux of eddy momentum into the tube. The second term in zonal component of the E-P flux equals the vertical component of the net pressure force on the tube. The vertical component of the E-P flux represents the x-projection of the net pressure force pushing against the isentropes. In addition to eddy forcing, the mean flow also experienced self-feedback of mean momentum by $\nabla \cdot \boldsymbol{J}$ and dissipative forces that are presented by $\overline{Q}$. Obviously, the basic flow will be intensified for the negative divergence of the E-P flux, while the positive divergence of E-P flux weakens the basic flow.

Using the local area averages instead of the density-weighted averages, we have the following equation

$$\frac{\partial}{\partial t}(\overline{\sigma}\overline{u}) = -\nabla \cdot \boldsymbol{J}_m - \nabla \cdot \boldsymbol{F}_m - \frac{\partial}{\partial t}(\overline{\sigma' u'}) + f(\overline{\sigma}\overline{v} + \overline{\sigma' v'}) + Q_m, \tag{14}$$

where

$$\boldsymbol{J}_m = \begin{bmatrix} \overline{u\sigma}\,\bar{u} + \dfrac{R}{R+c_p}\dfrac{\bar{p}}{g}\dfrac{\partial\bar{\varphi}}{\partial\theta} \\ \overline{u\sigma v} \\ -\dfrac{\bar{p}}{g}\dfrac{\partial\bar{\varphi}}{\partial x} \end{bmatrix}, \qquad \boldsymbol{F}_m = \begin{bmatrix} \overline{\sigma u'u'} + 2\bar{u}\,\overline{\sigma'u'} + \dfrac{1}{g}\dfrac{R}{R+c_p}\overline{p'\dfrac{\partial\varphi'}{\partial\theta}} \\ \overline{\sigma u'v'} + \bar{u}\,\overline{\sigma'v'} + \bar{v}\,\overline{\sigma'u'} \\ -\dfrac{1}{g}\overline{p'\dfrac{\partial\varphi'}{\partial x}} \end{bmatrix},$$

and

$$Q_m = \overline{\sigma F} - \frac{\partial}{\partial\theta}(\bar{\dot\theta}\bar{\sigma}\,\bar{u} + \overline{\sigma u'\dot\theta'} + \bar{u}\,\overline{\sigma'\dot\theta'} + \bar{\dot\theta}\,\overline{\sigma'u'}).$$

Equation(14) resembles Eq. (13) except that the temporal change of eddy momentum also contributes to the evolution of basic flow and that the E-P flux vector contains the eddy momentum transports by the basic flow $(\bar{u},\bar{v})$. Since we do not take any approximation in the derivation, Eqs. (13) and (14) may be applicable to small-amplitude and finite-amplitude disturbances.

III. WAVE-ACTIVITY CONSERVATION LAW

In this section, our goal is to obtain an ageostrophic expression of wave-activity conservative relation in view of Ertel's potential vorticity (EPV) in isentropic coordinates. The ageostrophic EPV equation is expressed as follows

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = \frac{1}{\sigma}\Big[\frac{\partial}{\partial x}\Big(G - \dot\theta\frac{\partial v}{\partial\theta}\Big) - \frac{\partial}{\partial y}\Big(F - \dot\theta\frac{\partial u}{\partial\theta}\Big) + q\frac{\partial\sigma\dot\theta}{\partial\theta}\Big], \tag{15}$$

where $q=\frac{\zeta}{\sigma}$ is the Ertel's potential vorticity, and $\zeta=f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ is the absolute vertical vorticity in isentropic coordinates. Taking small-amplitude approximation, the linearized primitive equations are given by

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} + u'\frac{\partial\bar{u}}{\partial x} + v'\frac{\partial\bar{v}}{\partial x} + \frac{\partial\varphi'}{\partial x} - v'\bar{\zeta} = F - \dot\theta'\frac{\partial\bar{u}}{\partial\theta}, \tag{16}$$

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{\partial y} + u'\frac{\partial\bar{u}}{\partial y} + v'\frac{\partial\bar{v}}{\partial y} + \frac{\partial\varphi'}{\partial y} + u'\bar{\zeta} = G - \dot\theta'\frac{\partial\bar{v}}{\partial\theta}, \tag{17}$$

$$\frac{\partial\sigma'}{\partial t} + \bar{u}\frac{\partial\sigma'}{\partial x} + \bar{v}\frac{\partial\sigma'}{\partial y} + \sigma'\Big(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y}\Big) + \frac{\partial u'\bar{\sigma}}{\partial x} + \frac{\partial v'\bar{\sigma}}{\partial y} = -\frac{\partial\dot\theta'\bar{\sigma}}{\partial\theta}, \tag{18}$$

$$\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x} + \bar{v}\frac{\partial q'}{\partial y} + u'\frac{\partial\bar{q}}{\partial x} + v'\frac{\partial\bar{q}}{\partial y} = Q', \tag{19}$$

where, $u'=u-\bar{u}$, $v'=v-\bar{v}$, $\dot\theta=\dot\theta'$, $\varphi'=\varphi-\bar{\varphi}$, $\sigma'=\sigma-\bar{\sigma}$, $\zeta'=\zeta-\bar{\zeta}$, $q'=q-\bar{q}$, $\bar{q}=\frac{\bar{\zeta}}{\bar{\sigma}}$, $\bar{\zeta}=f+\frac{\partial\bar{v}}{\partial x}-\frac{\partial\bar{u}}{\partial y}$ and $Q'=\frac{1}{\bar{\sigma}}\Big[\frac{\partial}{\partial x}\Big(G-\dot\theta'\frac{\partial\bar{v}}{\partial\theta}\Big)-\frac{\partial}{\partial y}\Big(F-\dot\theta'\frac{\partial\bar{u}}{\partial\theta}\Big)+\bar{q}\frac{\partial\dot\theta'\bar{\sigma}}{\partial\theta}\Big]$, the overbar denotes the basic states and the prime denotes the disturbances. It should be specially noted that the basic states chosen in this section may be any kind of climatic background fields, and are not confined to the local area averages defined by Eq. (10) or density-weighted averages defined by Eq. (11).

Multiplying Eq. (19) by q' yields the EPV enstrophy equation as follows

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}q'^2\Big) + \bar{u}\frac{\partial}{\partial x}\Big(\frac{1}{2}q'^2\Big) + \bar{v}\frac{\partial}{\partial y}\Big(\frac{1}{2}q'^2\Big) + u'q'\frac{\partial\bar{q}}{\partial x} + v'q'\frac{\partial\bar{q}}{\partial y} = q'Q'. \tag{20}$$

It can be easily shown that with small-amplitude assumption the EPV may be lineariezed as

$$\bar{\sigma}q' = \zeta' - \sigma'\bar{q} \qquad \text{or} \quad \bar{\sigma}^2 q' = \bar{\sigma}\zeta' - \sigma'\bar{\zeta}. \tag{21}$$

Thus, the zonal and meridional fluxes of eddy EPV may be expressed as follows

$$\bar{\sigma}^2 u'q' = \frac{\partial u'v'\bar{\sigma}}{\partial x} - \bar{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{2}u'^2\right) - v'\frac{\partial u'\bar{\sigma}}{\partial x} - \sigma' u'\bar{\zeta}, \tag{22}$$

$$\bar{\sigma}^2 v'q' = \bar{\sigma}\frac{\partial}{\partial x}\left(\frac{1}{2}v'^2\right) - \frac{\partial u'v'\bar{\sigma}}{\partial y} + u'\frac{\partial v'\bar{\sigma}}{\partial y} - \sigma' v'\bar{\zeta}. \tag{23}$$

The disturbance horizontal momentum equations are given by

$$\frac{\partial u'\sigma'}{\partial t} + \frac{\partial \bar{u}u'\sigma'}{\partial x} + \frac{\partial \bar{v}u'\sigma'}{\partial y} + u'\frac{\partial u'\bar{\sigma}}{\partial x} + u'\frac{\partial v'\bar{\sigma}}{\partial y} + u'\sigma'\frac{\partial \bar{u}}{\partial x} + v'\sigma'\frac{\partial \bar{v}}{\partial x} + \sigma'\frac{\partial \phi}{\partial x} - \sigma' v'\bar{\zeta}$$
$$= \sigma'\left(F - \dot{\theta}'\frac{\partial \bar{u}}{\partial \theta}\right) - v'\frac{\partial \dot{\theta}'\bar{\sigma}}{\partial \theta}, \tag{24}$$

$$\frac{\partial v'\sigma'}{\partial t} + \frac{\partial \bar{u}v'\sigma'}{\partial x} + \frac{\partial \bar{v}v'\sigma'}{\partial y} + v'\frac{\partial u'\bar{\sigma}}{\partial x} + v'\frac{\partial v'\bar{\sigma}}{\partial y} + u'\sigma'\frac{\partial \bar{u}}{\partial y} + v'\sigma'\frac{\partial \bar{v}}{\partial y} + \sigma'\frac{\partial \phi}{\partial y} + \sigma' u'\bar{\zeta}$$
$$= \sigma'\left(G - \dot{\theta}'\frac{\partial \bar{v}}{\partial \theta}\right) - v'\frac{\partial \dot{\theta}'\bar{\sigma}}{\partial \theta}. \tag{25}$$

Using the second order approximation

$$\sigma'\frac{\partial \phi}{\partial x} \approx -\frac{\partial}{\partial \theta}\left(\frac{p'}{g}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right), \tag{26}$$

$$\sigma'\frac{\partial \phi}{\partial y} \approx -\frac{\partial}{\partial \theta}\left(\frac{p'}{g}\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right), \tag{27}$$

and eliminating $\sigma' u'\bar{\zeta}$ in Eq. (22) by Eq. (25) and $\sigma' v'\bar{\zeta}$ in Eq. (23) by Eq. (24), we have

$$\bar{\sigma}^2 u'q' = \frac{\partial}{\partial x}(v'\sigma'\bar{u} + u'v'\bar{\sigma}) + \frac{\partial}{\partial y}\left(2v'\sigma'\bar{v} + u'\sigma'\bar{u} + v'^2\bar{\sigma} + \frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right) +$$
$$\frac{\partial}{\partial \theta}\left(-\frac{p'}{g}\frac{\partial \phi}{\partial y}\right) - \bar{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2\right) - \bar{u}\frac{\partial u'\sigma'}{\partial y} - \bar{v}\frac{\partial v'\sigma'}{\partial y} + \frac{\partial v'\sigma'}{\partial t} -$$
$$\left[\sigma'\left(G - \dot{\theta}'\frac{\partial \bar{v}}{\partial \theta}\right) - v'\frac{\partial \dot{\theta}'\bar{\sigma}}{\partial \theta}\right], \tag{28}$$

$$\bar{\sigma}^2 v'q' = \frac{\partial}{\partial x}\left(-2u'\sigma'\bar{u} - u'^2\bar{\sigma} - v'\sigma'\bar{v} - \frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial y}(-u'\sigma'\bar{v} - u'v'\bar{\sigma}) +$$
$$\frac{\partial}{\partial \theta}\left(\frac{p'}{g}\frac{\partial \phi}{\partial x}\right) + \bar{\sigma}\frac{\partial}{\partial x}\left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2\right) + \bar{u}\frac{\partial u'\sigma'}{\partial x} + \bar{v}\frac{\partial v'\sigma'}{\partial x} - \frac{\partial u'\sigma'}{\partial t} +$$
$$\sigma'\left(F - \dot{\theta}'\frac{\partial \bar{u}}{\partial \theta}\right) - u'\frac{\partial \dot{\theta}'\bar{\sigma}}{\partial \theta}. \tag{29}$$

Multiplying Eq. (20) by $\frac{\bar{\sigma}^2}{|\nabla\bar{q}|}$ and then substituting Eqs. (28) and (29) into Eq. (20), we obtain

$$\frac{\bar{\sigma}^2}{|\nabla\bar{q}|}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}q'^2\right) + \bar{u}\frac{\partial}{\partial x}\left(\frac{1}{2}q'^2\right) + \bar{v}\frac{\partial}{\partial y}\left(\frac{1}{2}q'^2\right)\right] +$$
$$\frac{\partial}{\partial x}\left[(v'\sigma'\bar{u} + u'v'\bar{\sigma})n_i + \left(-2u'\sigma'\bar{u} - u'^2\bar{\sigma} - v'\sigma'\bar{v} - \frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right)n_j\right] +$$
$$\frac{\partial}{\partial y}\left[\left(2v'\sigma'\bar{v} + u'\sigma'\bar{u} + v'^2\bar{\sigma} + \frac{R}{R+c_p}\frac{p'}{g}\frac{\partial \phi}{\partial \theta}\right)n_i + (-u'\sigma'\bar{v} - u'v'\bar{\sigma})n_j\right] +$$
$$\frac{\partial}{\partial \theta}\left[\left(-\frac{p'}{g}\frac{\partial \phi}{\partial y}\right)n_i + \left(\frac{p'}{g}\frac{\partial \phi}{\partial x}\right)n_j\right] + \bar{\sigma}\left[-\frac{\partial}{\partial y}\left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2\right)n_i + \frac{\partial}{\partial x}\left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2\right)n_j\right] +$$

$$\bar{u}\left(-\frac{\partial u'\sigma'}{\partial y}n_i+\frac{\partial u'\sigma'}{\partial x}n_j\right)+\bar{v}\left(-\frac{\partial v'\sigma'}{\partial y}n_i+\frac{\partial v'\sigma'}{\partial x}n_j\right)+\frac{\partial v'\sigma'}{\partial t}n_i-\frac{\partial u'\sigma'}{\partial t}n_j$$
$$=S_D, \tag{30}$$

where

$$n_i=\frac{1}{|\nabla\bar{q}|}\frac{\partial\bar{q}}{\partial x},\quad n_j=\frac{1}{|\nabla\bar{q}|}\frac{\partial\bar{q}}{\partial y}$$

and

$$S_D=\frac{1}{|\nabla\bar{q}|}q'\left[\bar{\sigma}\frac{\partial}{\partial x}\left(G-\dot{\theta}'\frac{\partial\bar{v}}{\partial\theta}\right)-\bar{\sigma}\frac{\partial}{\partial y}\left(F-\dot{\theta}'\frac{\partial\bar{u}}{\partial\theta}\right)+\bar{\zeta}\frac{\partial\dot{\theta}'\bar{\sigma}}{\partial\theta}\right]+$$
$$\left[\sigma'\left(G-\dot{\theta}'\frac{\partial\bar{v}}{\partial\theta}\right)-\frac{v'\partial\dot{\theta}'\bar{\sigma}}{\partial\theta}\right]n_i+\left[-\sigma'\left(F-\dot{\theta}'\frac{\partial\bar{u}}{\partial\theta}\right)+u'\frac{\partial\dot{\theta}'\bar{\sigma}}{\partial\theta}\right]n_j.$$

Following Plumb (1985; 1986) and Takaya (2001), if one assumes that the basic variables vary slowly in space and time, namely, $\frac{\bar{\sigma}^2}{|\nabla\bar{q}|}$, $\bar{u}$, $\bar{v}$, $\bar{\sigma}$, n_i, and n_j are the temporally and spatially slowly-varying functions, then Eq. (30) becomes

$$\frac{\partial A_s}{\partial t}+\nabla\cdot\boldsymbol{F}_s=S_D, \tag{31}$$

where $A_s=\frac{1}{|\nabla\bar{q}|}\left(\frac{1}{2}q'^2\bar{\sigma}^2+v'\sigma'\frac{\partial\bar{q}}{\partial x}-u'\sigma'\frac{\partial\bar{q}}{\partial y}\right)$ is the wave-activity density,

$$\boldsymbol{F}_s=\begin{pmatrix}\frac{1}{2}\frac{\bar{u}\bar{\sigma}^2q'^2}{|\nabla\bar{q}|}+(v'\sigma'\bar{u}+u'v'\bar{\sigma})n_i-\left[u'\sigma'\bar{u}+\frac{1}{2}\bar{\sigma}(u'^2-v'^2)+\frac{R}{R+c_p}\frac{p'}{g}\frac{\partial\phi}{\partial\theta}\right]n_j\\ \frac{1}{2}\frac{\bar{u}\bar{\sigma}^2q'^2}{|\nabla\bar{q}|}+\left[v'\sigma'\bar{v}+\frac{1}{2}\bar{\sigma}(v'^2-u'^2)+\frac{R}{R+c_p}\frac{p'}{g}\frac{\partial\phi}{\partial\theta}\right]n_i-(u'\sigma'\bar{v}+u'v'\bar{\sigma})n_j\\ \frac{p'}{g}\frac{\partial\phi}{\partial x}n_j-\frac{p'}{g}\frac{\partial\phi}{\partial y}n_i\end{pmatrix}$$

is the so-called 3-D flux of wave-activity density, and S_D is the source or sink term associated with the forcing and dissipation.

It is shown that the ageotrophic EPV enstrophy equation (31) has the conservative form when the forcing of friction and dabiatic heating vanishes. When integrated over a fixed region on whose boundary the normal component of $\boldsymbol{F}_s$ vanishes, the wave activity density is conservative and invariant. The wave-activity conservative relation may be applied to diagnose the propagation of wave activity and the evolution of local enstrophy that is associated with the development or destruction of weather system.

Plumb (1986) and Takaya (2001) examined the quasi-geotrosphic wave-activity conservative laws for ageostrophic small-amplitude. The wave-activity conservative relation obtained here is suitable for the ageostrophic small-amplitude disturbances because we have used the linearization approximation in the derivation of Eq. (31).

IV. DISCUSSION

Although the quasi-geostrophic theory of wave-basic flow interactions has been successfully applied to the operational forecasting, the coherent limitation of quasi-geostrophic approximation imposes some constraining conditions on the practical application of E-P flux. It is the limitation that encourages us to generalize the quasi-geostrophic theory of wave-basic flow interaction into the ageostrophic version.

In this work, we investigate the relation between 3-D E-P flux and two kinds of averaged flows and derive the wave-activity conservation law in the ageotrophic framework by using the primitive equations in isentropic coordinates. As what Takaya (2001) noted, the wave-activity density flux may not be particularly suitable for evaluating the exact local budget of wave-activity density because of several assumptions imposed in the derivation of wave-activity conservation law. Nevertheless, these assumptions seem qualitatively valid in the assessment based on observed and simulated data. The E-P flux and the wave-activity density flux are usefully diagnostic tools for illustrating propagations of disturbances and for judging where the wave enstrophy is converged, emitted or absorbed. One of advantages of the theory is that the E-P flux and wave-activity conservative law are easily calculated for observation data or model output data because all of terms are presented in terms of Eulerian quantities.

REFERENCES

Andrews, David G. (1983), Finite-amplitude Eliassen-Palm theorem in isentropic coordinates. *J. Atmos. Sci.*, **40**: 1877—1883.

Andrews, D. G. and McIntyre, M. E. (1976), Planetary waves in horizontal and vertical shear: the generalized Eliassen-Palm relation and the mean zonal acceleration, *J. Atmos. Sci.*, **33**: 2031—2048.

Dunkerton, T., Esu, C.-P. F. and McIntyre, M. E. (1981), Some Eulerian and Lagrangian diagnostics for a model stratospheric warming, *J. Atmos. Sci.*, **38**: 819—843.

Edmon, H. J., Hoskins, B. J. and McIntyre, M. E. (1980), Eliassen-Palm cross sections for the troposphere, *J. Atmos. Sci.*, **37**: 2600—2616.

Eliassen, A. and Palm, E. (1961), On the transfers of energy in stationary mountain waves, *Geofys. Publ.*, **22**: 1—23.

Gao Shouting, Tao Shiyan and Ding Yihui (1990), The generalized E-P flux of wave-meanflow interactions, *Science in China* (Series B), **33**: 704—715.

Haynes, P. H. (1988), Forced, dissipative generalizations of finite-amplitude wave activity conservation relations for zonal and nonzonal basic flows, *J. Atmos. Sci.*, **45**: 2352—2362.

Pfeffer, R. L. (1992), A study of eddy-induced fluctuations of the zonal-mean wind using conventional and transformed Eulerian diagnostics, *J. Atmos. Sci.*, **49**: 1036—1050.

Plumb, R. A. (1986), Three-dimensional propagation of transient quasi-geostrophic eddies and its relationship with the eddy forcing of the time-mean flow, *J. Atmos. Sci.*, **43**: 1657—1678.

Takaya, K. and Nakamura, H. (2001), A formulation of a phase-independent wave-activity flux for stationary and migratory quasigeostrophic eddies on a zonally varying basic flow, *J. Atmos. Sci.*, **58**: 608—627.

Tung, K. K. (1986), Nongeostrophic theory of zonally averaged circulation, Part I: Formulation, *J. Atmos. Sci.*, **43**: 2600—2618.

Yang, H., Tung, K. K. and Olaguer, E. (1990), Nongeostrophic theory of zonally averaged circulation. Part II: Eliassen-Palm flux divergence and isentropic mixing coefficient, *J. Atmos. Sci.*, **47**: 215—241.